

ERASMUS UNIVERSITY ROTTERDAM
Information concerning the Entrance examination
Mathematics level 3 for Econometrics

General information

Available time: 3 hours (180 minutes).

In all your answers, give a complete solution where you show all the required steps and substitutions that lead to your answer. The quality and completeness of your detailed solutions determine the points you will get. You should end an exercise with a conclusion or an answer.

The use of a calculator is not permitted.

Content information

In the exam you may find questions regarding the following topics:

A: NUMBERS AND RULES OF CALCULATION

1. The sets \mathbb{N} , \mathbb{Z} , \mathbb{Q} , \mathbb{R} and the operations addition, subtraction, multiplication and division.

$$\begin{aligned} \text{Natural numbers: } \mathbb{N} &= \{0, 1, 2, 3, \dots\} \\ \text{Integers: } \mathbb{Z} &= \{\dots, -2, -1, 0, 1, 2, \dots\} \\ \text{Rational numbers: } \mathbb{Q} &= \left\{ \frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{N}, b \neq 0 \right\} \end{aligned}$$

2. Absolute value $|x|$ and simple graphs corresponding with absolute value.

B: ALGEBRAIC SKILLS

1. Fractions

$$\begin{aligned} \bullet \frac{1}{a} + \frac{1}{b} &= \frac{b}{a \cdot b} + \frac{a}{a \cdot b} = \frac{a+b}{a \cdot b} \\ \bullet \frac{a}{b} + c &= \frac{a}{b} + \frac{b \cdot c}{b} = \frac{a + b \cdot c}{b} \\ \bullet \frac{a}{b} + \frac{c}{d} &= \frac{a \cdot d}{b \cdot d} + \frac{b \cdot c}{b \cdot d} = \frac{a \cdot d + b \cdot c}{b \cdot d} \\ \bullet a \cdot \frac{b}{c} &= \frac{a \cdot b}{c} = \frac{a}{c} \cdot b = a \cdot b \cdot \frac{1}{c} \\ \bullet \frac{a}{b} \cdot \frac{c}{d} &= \frac{a \cdot c}{b \cdot d} \\ \bullet \frac{a}{\frac{b}{c}} &= a \cdot \frac{c}{b} = \frac{a \cdot c}{b} \end{aligned}$$

2. Special products

- $(a \pm b)^2 = a^2 \pm 2ab + b^2$
- $(a + b)(a - b) = a^2 - b^2$

3. Factorization of a polynomial of second degree (quadratic function) into linear terms.

For example:

- $f(x) = x^2 - 2x - 15$ can be factorized as follows: $f(x) = (x - 5)(x + 3)$
- $f(x) = x^2 - 2x$ can be factorized as follows: $f(x) = x(x - 2)$

4. The power rules and corresponding rules for logarithms. Use of rational and negative exponents.

$$\begin{aligned}
 a^p \cdot a^q &= a^{p+q} \\
 \frac{a^p}{a^q} &= a^{p-q} \\
 (a^p)^q &= a^{pq} \\
 (ab)^p &= a^p b^p \\
 \left(\frac{a}{b}\right)^p &= \frac{a^p}{b^p} \\
 a^{-p} &= \frac{1}{a^p} \\
 a^{\frac{1}{p}} &= \sqrt[p]{a} \\
 {}^g\log a^p &= p \cdot {}^g\log a \\
 {}^g\log a &= \frac{\log a}{\log g} = \frac{\ln a}{\ln g} \\
 {}^g\log a + {}^g\log b &= {}^g\log ab \\
 {}^g\log a - {}^g\log b &= {}^g\log \frac{a}{b}
 \end{aligned}$$

Remark: log is the commonly used notation for ${}^{10}\log$

5. Application of the items mentioned in B1 through 4

- *through substitution of numbers*
- *through substitution of expressions*
- *by reformulation and reduction of expressions*

C: STANDARD FUNCTIONS

1. Properties of the following standard functions:

- polynomials

In particular:

the linear function $y = ax + b$

the quadratic function $y = ax^2 + bx + c$

- the rational function $y = \frac{ax+b}{cx+d}$
- power functions $y = c \cdot x^n$

In particular the function $y = \sqrt{x}$

- exponential functions $y = g^x$ and their inverse functions $y = {}^g\log x$

- the function $y = e^x$ and its inverse function $y = \ln x$
- the functions $y = \sin x$ and $y = \cos x$

In particular the following properties:

- The values of $\sin(x)$ and $\cos(x)$ at $x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ and $\frac{\pi}{2}$
 - Formulas for the following expressions in $\sin(x)$, and similarly for $\cos(x)$: $\sin(-x)$, $\sin(\pi - x)$, $\sin(\frac{\pi}{2} - x)$, $\sin(x \pm y)$, $\sin(x) \pm \sin(y)$
 - $\sin^2(x) + \cos^2(x) = 1$
2. Sketch the graphs of the standard functions and use the notions domain (all possible values of x for which the function is defined), range (the set of all values of y obtained as x varies in the domain), zeroes, ascending, descending function and asymptotic behaviour.

The function $y = 2^x$ has a horizontal asymptote $y = 0$, the function $y = \ln x$, which is only defined if $x > 0$, has a vertical asymptote $x = 0$.

3. Apply transformations on graphs such as shifting and stretching, and describe the link between the transformation and the alteration of the corresponding function.
- translation of $f(x)$ along the vector (a, b) results in $y = f(x - a) + b$
 - multiplication of $f(x)$ with a with respect to the x -axis results in $y = a \cdot f(x)$
 - multiplication of $f(x)$ with b with respect to the y -axis results in $y = f(x/b)$

For example: argue that the domain of $y = \sqrt{3x - 6}$ is equal to $x \geq 2$.

For example: sketch the graph of $f(x) = 2 + 3\sqrt{x - 6}$, where $x \geq 6$, based on a sketch of the graph of \sqrt{x}

For example: Sketch the graphs of $y = a \sin(bx + c) + d$ and $y = a \cos(bx + c) + d$, using the notions period ($= 2\pi/b$), amplitude ($= a$), equilibrium value ($= d$) and shift to the right ($= -c/b$).

D: EQUATIONS, SYSTEMS OF EQUATIONS AND INEQUALITIES

1. Solving equations using the following general rules

- $a \cdot b = 0 \Leftrightarrow a = 0$ or $b = 0$
- $a \cdot b = a \cdot c \Leftrightarrow a = 0$ or $b = c$
- $\frac{a}{b} = c \Leftrightarrow a = b \cdot c$ and $b \neq 0$
- $\frac{a}{b} = \frac{c}{d} \Leftrightarrow a \cdot d = b \cdot c$ and $b, d \neq 0$
- $a^2 = b^2 \Leftrightarrow a = b$ or $a = -b$

2. Solve equations with polynomials using standard algorithms

- *Linear equations*
- *Quadratic equations of the form $ax^2 + b = 0$ or $ax^2 + bx = 0$*

- Solve general quadratic equations of the form $ax^2 + bx + c = 0$ by means of factorization or by means of the so-called “quadratic formula”:

The solutions of $ax^2 + bx + c = 0$ are given by $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- Linear or quadratic equations with parameters

- Solve equations of the type:

$$\begin{aligned} x^a = b &\Rightarrow x = b^{\frac{1}{a}} = \sqrt[a]{b} \\ g^x = a &\Rightarrow x = {}^g\log a = \frac{\log a}{\log g} = \frac{\ln a}{\ln g} \\ {}^g\log x = a &\Rightarrow x = g^a \end{aligned}$$

- The solution of $\sin x = 0$, $\cos x = 0$, $\sin t = \sin u$, and $\cos t = \cos u$:

$$\begin{aligned} \sin(x) = 0 &\Rightarrow x = k\pi && k \in \mathbb{Z} \\ \cos(x) = 0 &\Rightarrow x = \pi/2 + k\pi && k \in \mathbb{Z} \\ \sin(x) = \sin(y) &\Rightarrow x = y + 2k\pi \text{ or } x = \pi - y + 2k\pi && k \in \mathbb{Z} \\ \cos(x) = \cos(y) &\Rightarrow x = y + 2k\pi \text{ or } x = -y + 2k\pi && k \in \mathbb{Z} \end{aligned}$$

- Use substitution to solve equations

For example: the equation $x^4 - 4x^2 + 3 = 0$ has the following roots: $x = \pm 1$ and $x = \pm\sqrt{3}$.

- Solve equations of the type $f(x) = g(x)$ algebraically
- Solve equations of the type $|f(x)| = g(x)$ algebraically
- Solve a system of two linear equations with two unknowns.

For example:

$$\begin{cases} 2x + 2y = 2 \\ 3x + y = 5 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = -1 \end{cases}$$

- Solve a system of three linear equations with three unknowns.
- Solve a system of two non-linear equations with two unknowns.

For example:

$$\begin{cases} x^2 + y^2 = 100 \\ y = \frac{1}{2}x^2 - 26 \end{cases} \Rightarrow (x, y) = (6, -8), (-6, -8), (8, 6), (-8, 6)$$

- Solve a system of three non-linear equations with three unknowns.
- Solve inequalities of the type $f(x) \geq g(x)$ by first solving $f(x) = g(x)$ algebraically and subsequently using a sketch of both graphs.

For example: When solving $x^2 - 3x - 10 > -x + 5$, we first compute the solutions of $x^2 - 3x - 10 = -x + 5$, i.e. $x = 5$ or $x = -3$. A sketch of the graphs supplies the answer to the problem: $x < -3$ or $x > 5$.

E: CALCULUS

1. Recognize and use the different symbols for the derivative of a function.

$f'(x)$, $\frac{dy}{dx}$, $\frac{df(x)}{dx}$ and $\frac{df}{dx}(x)$ all denote the same notion.

2. Compute the derivative of the sum, the product and the quotient of standard functions, as described in item C1. Use the chain rule to compute the derivative of combinations of these functions.

Some examples:

sum: $f(x) = \sin(x) + 5 \cos(x) \Rightarrow f'(x) = \cos(x) - 5 \sin(x)$

product: $f(x) = (x^2 - 3x + 5) \ln x \Rightarrow f'(x) = (2x - 3) \ln x + (x^2 - 3x + 5) \cdot \frac{1}{x}$

quotient: $f(x) = \frac{x^2}{5x + 3} \Rightarrow f'(x) = \frac{(5x + 3) \cdot 2x - x^2(5)}{(5x + 3)^2} = \frac{5x^2 + 6x}{(5x + 3)^2}$

chain rule: $f(x) = e^{4x^2 - 5} \Rightarrow f'(x) = 8xe^{4x^2 - 5}$

3. Use the differential quotient ($\Delta y / \Delta x$) as indication of the local change of a function.
4. Determine on which intervals a function is constant (derivative = 0), ascending (derivative > 0) or descending (derivative < 0).
5. Determine (by inspecting the sign of the derivative) whether a function has maximum and/or minimum values, and calculate the coordinates of these extreme values.

Example: the derivative of the function $f(x) = x^2 - 2x - 15$ is equal to 0 when $x = 1$. For every $x < 1$ the function is decreasing, for every $x > 1$ the function increasing, so the function has a minimum at $x = 1$. The coordinates of this minimum are $(1, -16)$.

6. Determine the second derivative $f''(x)$ of the sum, the product and the quotient of standard functions as described in item C1 and of combinations of these functions.
7. Determine (by inspecting the sign of the second derivative) whether a function has points of inflection, and calculate the coordinates of such inflection points.

- *The function $f(x) = (x - 1)^3 + 4$ in the example above has a point of inflection at $x = 1$. The coordinates are $(1, 4)$.*
- *The second derivative of the function $f(x) = (x - 1)^4 + 4$ is equal to $f''(x) = 12(x - 1)^2$. This second derivative is equal to 0 only if $x = 1$. For every $x < 1$ the function f satisfies $f''(x) > 0$ and for every $x > 1$ the function also satisfies $f''(x) > 0$. Hence, this function does not have any point of inflection.*

F: STRAIGHT LINES, TANGENT LINES AND PERPENDICULAR INTERSECTION

1. Determine the formula of a straight line either in case two of its points are given, or in case a single point and the slope of the line are given.
2. Know the conditions for two parallel straight lines and two perpendicular straight lines.

The two lines $y = ax + b$ and $y = cx + d$ are parallel when $a = c$. The lines are perpendicular when $ac = -1$.

3. Give the formulas for the tangent line and the normal line (i.e. the line perpendicular to the tangent line) at a given point on the graph of (combinations of) standard functions.

Example: the function $f(x) = x^2 - 2x - 15$ satisfies $f(6) = 9$ and $f'(6) = 10$. The tangent line of the graph of $f(x)$ at $x = 6$ is described by $y = 10x - 51$. The normal line at the point with x -coordinate equal to 6 is given by $y = -\frac{1}{10}x + \frac{96}{10}$.

4. For two given functions, determine their points of tangency.

Example: Consider the functions $f(x) = x - \ln(x)$ and $g_p(x) = px$. Calculate the value of p for which these functions have a point of tangency and calculate the coordinates of the point of tangency. Answer: the x -coordinate of the point of tangency satisfies the following 2 conditions:

$$\begin{cases} f(x) = g_p(x) \\ f'(x) = g'_p(x) \end{cases} \Rightarrow \begin{cases} x - \ln(x) = px \\ 1 - \frac{1}{x} = p \end{cases}$$

Solving the system of 2 equations yields $x = e$ and $p = 1 - \frac{1}{e}$. The corresponding y -coordinate is equal to $e - 1$.

5. Determine points of perpendicular intersection of two given functions

G: INTEGRATION

1. Recognize and use the notations $\int f(x) dx$ and $\int_a^b f(x) dx$.

Examples:

- $\int 6x dx = 3x^2 + C$
- $\int_1^2 6x dx = [3x^2]_1^2 = 12 - 3 = 9$

2. Calculate an integral exactly in case the integrand is a standard function (see C1), or a combination of standard functions.

Examples:

- $\int_0^1 (6x^2 + 8x - 7) dx = [2x^3 + 4x^2 - 7x]_0^1 = -1$

- $\int_0^1 3\sqrt{x} \, dx = \left[2x^{\frac{3}{2}} \right]_0^1 = 2$
- $\int_4^9 \frac{1}{\sqrt{x}} \, dx = \left[2\sqrt{x} \right]_4^9 = 2$
- $\int_2^3 \frac{1}{x} \, dx = \left[\ln|x| \right]_2^3 = \ln 3 - \ln 2 = \ln \frac{3}{2}$

3. Calculate integrals using substitutions.

Examples:

- $\int_1^2 \sqrt{4x-1} \, dx = \left[\frac{1}{6}(4x-1)^{\frac{3}{2}} \right]_1^2 = \frac{1}{6}(7\sqrt{7} - 3\sqrt{3}) = \frac{7}{6}\sqrt{7} - \frac{1}{2}\sqrt{3}$
- $\int_3^7 \frac{2}{\sqrt{4x-3}} \, dx = \left[\sqrt{4x-3} \right]_3^7 = 2$
- $\int_1^2 \frac{6x}{3x^2-1} \, dx = \left[\ln|3x^2-1| \right]_1^2 = \ln 11 - \ln 2 = \ln \frac{11}{2}$
- $\int_1^2 x^2 e^{x^3-5} \, dx = \left[\frac{1}{3} e^{x^3-5} \right]_1^2 = \frac{1}{3} e^3 - \frac{1}{3} e^{-4}$

4. Note: you are allowed to use the technique called Integration by Parts, but this technique is not necessary for solving the exam problems.
5. Use an integral when calculating areas under a given curve and volumes of a solid of revolution:

- Given a function $y = f(x)$ on an interval $[a, b]$, the area between $f(x)$, the x -axis and the lines $x = a$ and $x = b$ is equal to

$$\int_a^b |f(x)| \, dx$$

- Given 2 functions $f(x) \geq g(x) \geq 0$ on an interval $[a, b]$, the area between $f(x)$ and $g(x)$ and the lines $x = a$ and $x = b$ is equal to

$$\int_a^b (f(x) - g(x)) \, dx$$

- Given a function $y = f(x)$ on an interval $[a, b]$. Let R denote the region bounded by this curve, by the x -axis and by the vertical lines $x = a$ and $x = b$. When we rotate R about the x -axis, we get a solid of revolution; the volume of this 3-dimensional object is equal to

$$\int_a^b \pi (f(x))^2 \, dx$$

- Given 2 functions $f(x) \geq g(x) \geq 0$ on an interval $[a, b]$. When we rotate these curves about the x -axis, we get the surfaces of two solids of revolution; the volume between these 3-dimensional objects is equal to

$$\int_a^b \pi ((f(x))^2 - (g(x))^2) \, dx$$

- Given a function $x = f(y)$ on an interval $[a, b]$. Let R denote the region bounded by this curve, by the y -axis and by the horizontal lines $y = a$ and $y = b$. When we rotate R about the y -axis, we get a solid of revolution; the volume of this 3-dimensional object is equal to

$$\int_a^b \pi(f(y))^2 dy$$